THE VOLTERRA SYSTEM AND TOPOLOGY OF THE ISOSPECTRAL VARIETY OF ZERO-DIAGONAL JACOBI MATRICES

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Let us consider a symplectic manifold (X^{2n},ω) and an integrable system with Hamiltonian H and involutive integrals $F_1=H,F_2,\dots,F_n$. Let $X_F\subset X$ be a submanifold defined by equations $F_1=c_1,\dots,F_n=c_n$. This submanifold is called a level surface of integrals. The well-known Liouville-Arnold theorem [1] says that if X_F is compact and connected, then it is a torus. This makes investigating the topology of X_F trivial. However, it turns out that in some important examples of integrable systems such a submanifold is compact but its topology is quite complicated. This is due to the fact that in these examples X has points where either H is singular or ω is singular or degenerate, and the Liouville-Arnold theorem does not apply. However, in some of these situations it is possible to define the corresponding flow on the whole submanifold X_F and successfully use it to investigate the topology of X_F.

The first example of this kind was described by C. Tomei in his paper [2], where the topology of the isospectral variety of Jacobi matrices (i.e. real three-diagonal symmetric matrices) is investigated. This is a level surface of integrals of the (open) Toda lattice. Using the Toda flow Tomei computed the Euler characteristic of this variety. Two years later D. Fried [3] discovered that the stable and unstable stratifications defined by the Toda flow are cell complexes. Using this fact he found the cohomology ring of this isospectral variety by direct calculation.

Our goal is to investigate topology of the isospectral variety of real zero-diagonal Jacobi matrices. Let us consider the variety M_k of all $(k\times k)$-matrices L of the form $L_{ij}=c_i\delta_{i+1,j}+c_{i-1}\delta_{i-1,j}$ with fixed spectrum.

Proposition 1. *a) The eigenvalues of such a matrix are of the form 0 (if k is odd), $\pm\lambda_1,\pm\lambda_2,\dots$ b) If all eigenvalues are distinct then M_k is a compact smooth manifold and its topology does not depend on the eigenvalues.*

Proof of a) is clear. The proof of b) is analogous to Tomei's proof of the analogous statement for the isospectral variety of Jacobi matrices [2].

The manifold M_k is a level surface of integrals of the (open) Volterra system $\dot c_i=\frac{1}{2}c_i(c_{i+1}^2-c_{i-1}^2)$, where $c_0=c_k=0$. Usually this system is written in terms of the variables $u_i=c_i^2$. It is well-known that the Volterra system can be written in the Lax form $\dot L=[L,A(L)]$. It follows from this Lax form that the Volterra flow preserves the spectrum of L, i.e. the Volterra flow is a flow on M_k. It is well known [4] that the Volterra system is integrable. In the even ($k=2l$) case M_k has 2^l isomorphic connected components. Moreover, each component is isomorphic to the isospectral variety of Jacobi matrices studied by Tomei and Fried, see e.g. [5, 6]. The odd case ($k=2l+1$) is completely different, M_k is connected. It is not clear now how to compute the homology groups. The stable and unstable stratifications are not cell complexes in this case, and it is not possible to apply a direct approach similar to Fried's [3]. We should also say that in 1992 A. Bloch, R. Brockett and T. Ratiu [7] used the idea of the double bracket representation to show that the Toda flow is a gradient flow. It was shown in 1998 [8] that the Volterra flow is also a gradient flow. This leads us to the natural idea of finding the homology groups using the Morse complex. Unfortunately we cannot use the Morse complex for computing the homology groups because the stable and unstable stratifications are not transversal to each other. In the present paper we compute the Euler characteristic of M_k using the Volterra flow. The approach is similar to Tomei's [2].

Let $K=\frac{1}{4}\operatorname{diag}(1,2,3,\dots)$ and $f(L)=\operatorname{tr}KL^2$. It has been shown [8] that the (negative) gradient flow of $f(L)$ with respect to some metric coincides with the Volterra flow. Let us choose λ_i (see Proposition 1) in such a way that all $\lambda_i>0$ and $\lambda_i>\lambda_{i+1}$.

Proposition 2. *a) The critical points of $f(L)$ on M_{2l+1} are in one-to-one correspondence with triples $[j,s,\pi]$ consisting of a number j with $0\leqslant j\leqslant l$, an l-tuple $s=(s_1,\dots,s_l)$ with each s_i equal*

to 0 *or* 1*, and a permutation* $\pi \in S_l$*. b) The index of a critical point corresponding to* $[j, s, \pi]$ *is equal to the number of indexes* $0 \leqslant i \leqslant l-1$ *with* $i \neq j, j-1$ *and* $\pi(i) < \pi(i+1)$*, plus* 1 *if* $j \neq l$*.*

Proof. The critical points are equilibrium points of the Volterra flow. Using this fact and the explicit polynomial equations of M_k one can prove that the critical points are exactly those points with exactly l of the $2l$ values $c_1, \ldots, c_{2l}$ equal to zero and the additional property that $c_i \neq 0$ implies $c_{i-1} = c_{i+1} = 0$. Looking at spectrum of the corresponding matrices one obtains the description in the statement of the proposition. This proves a). The formula b) for the index may be obtained by studying the Hessian of $f(L)$.

Proposition 3. *a) The Euler characteristic of* M_{2l+1} *is equal to* $\chi(M_{2l+1}) = 2^{2l+2}(2^{l+2}-1)\frac{B_{l+2}}{l+2}$*, where* B_{l+2} *is a Bernoulli number. b) If we define* $\chi(M_1)$ *to be* 0*, then the exponential generating function is equal to* $-\tanh^2(2z)$*, i.e.,* $-\tanh^2(2z) = \sum_{l\geqslant 0} \chi(M_{2l+1})\frac{z^l}{l!}$*.*

Proof. Recall that an interval $[i, i+1]$ such that $\pi(i) < \pi(i+1)$ is called an ascent. The number of permutations of n elements with k ascents is called the Euler number $\left\langle {n \atop k} \right\rangle$, see [9]. Denote the number of ascents in π by $p(\pi)$. Let $\psi(n) = \sum_{m=0}^{n}(-1)^m \left\langle {n \atop k} \right\rangle$, then formula (7.56) in [9] implies that $1+\tanh z = \sum_{n\geqslant 0}\psi(n)\frac{z^n}{n!}$. Proposition 2 implies that the Euler characteristic $\chi(M_{2l+1})$ is equal to $2^l \sum\limits_{j=1}^{l-1} \binom{l}{j} \sum_{\pi_1\in S_j, \pi_2 \in S_{l-j}} (-1)^{p(\pi_1)+p(\pi_2)+1} = -2^l \sum\limits_{j=1}^{l-1} \binom{l}{j} (\sum_{\pi_1 \in S_j}(-1)^{p(\pi_1)})(\sum_{\pi_2\in S_{l-j}}(-1)^{p(\pi_2)}) = -2^l \sum_{j=1}^{l-1}\binom{l}{j}\psi(j)\psi(l-j)$. This implies b). Using the expansion of $\tanh z$ and the formula $\tanh' z = 1 - \tanh^2 z$, one obtains the formula from a).

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